

1.6.14 Application questions

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1. Some systems are modeled using continuous signals, while others are modeled using discrete signals. A continuous signal can take on any value at any point in time, whereas a discrete signal is represented by distinct, separate values sampled at specific time intervals. In practice, discrete signals often arise from periodically sampling continuous signals, such as those from sensors or other input devices.

For example, in digital audio, sound is typically sampled at 44.1 kHz, meaning air pressure is measured every $22\frac{2}{3} \mu\text{s}$ (microseconds). This sampling rate is chosen based on the Nyquist-Shannon sampling theorem, which states that to accurately reconstruct a signal, the sampling rate must be at least twice the highest frequency present in the signal. Since the highest frequency most humans can hear is around 20 kHz (just over five octaves above Middle A), the sampling rate of 44.1 kHz ensures that all audible frequencies can be faithfully reproduced without distortion.

Thus, discrete signals, like those used in digital audio, allow continuous phenomena—such as sound waves—to be represented in a format that can be processed and stored by digital systems.

One tool useful for modelling discrete signals is the z -transform, which helps analyze and manipulate signals in the frequency domain. If a discrete signal is $x = (x[0], x[1], x[2], \dots)$, the z -transform of this signal is defined as

$$\mathcal{Z}(x)(z) = X(z) = \sum_{k=0}^{\infty} x[k]z^{-k}.$$

In general, it is very difficult to calculate this infinite sum, but in certain cases, such as for specific types of sequences, simplifications can be made.

For example, if the discrete signal is a geometric sequence given by

$$\gamma^n = (1, \gamma, \gamma^2, \gamma^3, \dots)$$

where γ is a complex number, the z -transform becomes

$$\mathcal{Z}(\gamma^n)(z) = \sum_{k=0}^{\infty} \gamma^k z^{-k}.$$

This sum does not converge for all values of z . What is the relationship between $|\gamma|$ and $|z|$ for $\mathcal{Z}(\gamma^n)(z)$ to have a finite value, and what is that finite value?

2. In Question 1, suppose that $\gamma = 0.5 + 0.5j$, so

$$(0.5+0.5j)^n = (1, 0.5+0.5j, 0.5j, -0.25+0.25j, -0.25, -0.125-0.125j, -0.125j, 0.0625-0.0625j, 0.0625, \dots).$$

Find $\mathcal{Z}((0.5 + 0.5j)^n)(2 - j)$ as a complex number in the rectangular representation, $\alpha + \beta j$.

Aside: Vinyl records are an analog format, meaning they capture and reproduce continuous sound waves, while compact discs (CDs) are digital, storing discrete samples at a 44.1 kHz rate. Vinyl naturally comes with its own built-in “character,” including subtle distortions, compression, and the occasional snap, crackle, and pop—yes, like a breakfast cereal, but more expensive. With a dynamic range of around 60-70 dB, vinyl is more prone to wear and tear, and offers a more limited sound experience compared to CDs, which boast a wider dynamic range (up to 96 dB) and are free from the background hiss and degradation that vinyl inevitably collects like a lint roller. CDs also follow the Nyquist-Shannon theorem, ensuring they reproduce all audible frequencies with mathematical precision, like good sound engineers intended.

Ironically, all these vinyl imperfections are lovingly described by aficionados as producing a euphemistically “warmer” sound. What they actually mean is: they prefer their music filtered through a layer of distortion, noise, and vinyl-induced dynamic squashing. “Warmer” sounds inviting, when really it’s just a nice way to say, “We like our audio flawed, but in a charming way.” Amusingly, there are even digital filters designed to add these imperfections back into digital music, so you can have that “vintage” vinyl feel without the inconvenience of owning a turntable or a collection of fragile plastic discs.

When vinyl enthusiasts are asked to tell the difference between actual vinyl and digitally filtered music, things can get fuzzy—much like the audio they love. The moral of the story? Engineers must not only strive to achieve the best possible solutions based on objective, measurable factors (like low noise, low distortion, and high fidelity), but also to remember that sometimes, clients are chasing the subjective pleasure of imperfection. After all, nothing says “perfect sound” quite like a few charming pops and a dash of harmonic distortion, just do so with a digital filter instead of paying for and maintaining an expensive hi-fi audio system.

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